

# SHARP GAUSSIAN ESTIMATES FOR SCHRÖDINGER HEAT KERNELS: $L^p$ INTEGRABILITY CONDITIONS

KRZYSZTOF BOGDAN, JACEK DZIUBAŃSKI, AND KAROL SZCZYPKOWSKI

**ABSTRACT.** We give new sufficient conditions for comparability of the fundamental solution of the Schrödinger equation  $\partial_t = \Delta + V$  with the Gauss-Weierstrass kernel and show that local  $L^p$  integrability of  $V$  for  $p > 1$  is not necessary for the comparability.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $d = 1, 2, \dots$ . We consider the Gauss-Weierstrass kernel,

$$g(t, x, y) = (4\pi t)^{-d/2} e^{-|y-x|^2/(4t)}, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

It is well known that  $g$  is a time-homogeneous probability transition density. For a function  $V$  we let  $G$  be the Schrödinger perturbation of  $g$  by  $V$ , i.e., the fundamental solution of  $\partial_t = \Delta + V$ , determined by the following Duhamel or perturbation formula for  $t > 0$ ,  $x, y \in \mathbb{R}^d$ ,

$$G(t, x, y) = g(t, x, y) + \int_0^t \int_{\mathbb{R}^d} G(s, x, z) V(z) g(t-s, z, y) dz ds.$$

Under appropriate assumptions on  $V$ , the definition of  $G$  may be given by the Feynmann-Kac formula [5, Section 6], the Trotter formula [27, p. 467], the perturbation series [5], or by means of quadratic forms on  $L^2$  spaces [10, Section 4]. In particular the assumption  $V \in L^p(\mathbb{R}^d)$  with  $p > d/2$  was used by Aronson [2], Zhang [27, Remark 1.1(b)] and by Dziubański and Zienkiewicz [11]. Aizenman and Simon [1, 21] proposed functions  $V(z)$  from the Kato class, which contains  $L^p(\mathbb{R}^d)$  for every  $p > d/2$  [1, Chapter 4], [9, Chapter 3, Example 2]. An enlarged Kato class was used by Voigt [23] in the study of Schrödinger semigroups on  $L^1$  [23, Proposition 5.1]. For time-dependent perturbations  $V(u, z)$ , Zhang [24, 26] introduced the so-called parabolic Kato condition. It was then generalized and employed by Schnaubelt and Voigt [20], Liskevich and Semenov [17], Liskevich, Vogt and Voigt [18], and Gulisashvili and van Casteren [13].

We say that  $G$  has sharp Gaussian estimates if  $G$  is comparable with  $g$ , at least in bounded time (see below for details). A sufficient condition for the sharp Gaussian estimates was given by Zhang in [27]. As noted in [27, Remark 1.2(c)], the condition may be stated in terms of the *bridges* of  $g$ . Bogdan, Jakubowski and Hansen [5, Section 6] and Bogdan, Jakubowski and Sydor [6] gave analogous conditions for general transition densities.

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Given a real-valued Borel measurable function  $V$  on  $\mathbb{R}^d$  we ask if there are numbers  $0 < c_1 \leq c_2 < \infty$  such that the following two-sided bounds hold,

$$(1) \quad c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad t > 0, \ x, y \in \mathbb{R}^d.$$

We also ponder a weaker property—if for a given  $T \in (0, \infty)$ ,

$$(2) \quad c_1 \leq \frac{G(t, x, y)}{g(t, x, y)} \leq c_2, \quad 0 < t \leq T, \ x, y \in \mathbb{R}^d.$$

We call (1) and (2) *sharp Gaussian estimates* or bounds, respectively *global* and *local* in time. We observe that the inequalities in (1) and (2) are stronger than plain *Gaussian estimates*:

$$c_1 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t\varepsilon_1}} \leq G(t, x, y) \leq c_2 (4\pi t)^{-d/2} e^{-\frac{|y-x|^2}{4t\varepsilon_2}},$$

where  $0 < \varepsilon_1, c_1 \leq 1 \leq \varepsilon_2, c_2 < \infty$ . We note in passing that Berenstein proved the Gaussian estimates for  $V \in L^p$  with  $p > d/2$  (see [16]), Simon [21, Theorem B.7.1] resolved them for  $V$  in the Kato class, Zhang used sub-parabolic Kato class for the same end in [26] and the so-called 4G inequality was used by Bogdan and Szczypkowski in [7]. For further discussion we refer the reader to [17], [18], [19], [27], and the Introduction in [7].

It is difficult to explicitly characterize (1) and (2), especially for those  $V$  that take on positive values. Arsen'ev proved (2) for  $V \in L^p + L^\infty$  with  $p > d/2$ ,  $d \geq 3$ , and van Casteren [22] proved it for  $V$  in the intersection of the Kato class and  $L^{d/2} + L^\infty$  for  $d \geq 3$  (see [19]). Arsen'ev also obtained (1) for  $V \in L^p$  with  $p > d/2$  under additional smoothness assumptions (see [16]). Zhang [27, Theorem 1.1] and Milman and Semenov [19, Theorem 1C, Remark (2)] gave sufficient supremum-integral conditions for (2) and (1) for signed  $V$  and characterized (1) for  $V \leq 0$ . Their results left open certain natural questions about the class of admissible functions  $V$ , especially for dimensions  $d \geq 4$ . We were particularly motivated by the question of Liskevich and Semenov about the connection of the sharp Gaussian estimates, the potential-boundedness and the  $L^{d/2}$  integrability condition, cf. [16, Remark (3), p. 602]. In this work we use potential-boundedness (7) and bridges potential-boundedness of  $V$  to study the connection of the sharp Gaussian estimates and the  $L^p$  integrability, disregarding the Kato condition. In Theorem 2.9 below we give new sufficient conditions for the sharp Gaussian estimates, which help verify that  $L^p$  integrability is not necessary for (1) or (2). Namely, for  $d \geq 3$  we present in Corollary 3.4 functions  $V$  such that (1) holds but  $V \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L^p_{loc}(\mathbb{R}^d)$ . Our examples are highly anisotropic because they are constructed from tensor products, and to study them we crucially use factorization of the Gauss-Weierstrass kernel. Before discussing the present results we should mention our more recent paper [4], where we give a new characterization of (1) and resolve the question of Liskevich and Semenov. Both papers grew out from our work on this question but contain different observations. In fact [4] uses the preliminary results stated in this Introduction, apart from which the two papers have no overlap.

The structure of the remainder of the paper is the following. Below in this section we give definitions and preliminaries, and organize the relevant results existing in the literature. In particular in Lemma 1.1 we present

characterizations of (1) and (2) for  $V \leq 0$ . In Theorem 2.9 of Section 2 we propose new sufficient conditions for (1) and (2), with emphasis on those functions  $V$  which factorize as tensor products. In Section 3 we prove Corollary 3.4 and give examples which illustrate and comment our results. In Section 4 we give supplementary details.

Let  $\mathbb{N} = \{1, 2, \dots\}$ ,  $f^+ = \max\{0, f\}$  and  $f^- = \max\{0, -f\}$ . All the considered functions  $V: \mathbb{R}^d \rightarrow [-\infty, \infty]$  are assumed Borel measurable.

Here is a quantity to characterize (1) and (2):

$$S(V, t, x, y) = \int_0^t \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t-s, z, y)}{g(t, x, y)} |V(z)| dz ds, \quad t > 0, x, y \in \mathbb{R}^d.$$

In what follows we often abbreviate  $S(V)$ . The motivation for using  $S(V)$  comes from [27, Lemma 3.1 and Lemma 3.2] and from [5, (1)].

In the next two results we compile [27, Theorem 1.1] and observations from [5] and [6] to give conditions for the sharp Gaussian estimates. For completeness, the proofs are given in Section 4.

**Lemma 1.1.** *If  $V \leq 0$ , then (1) is equivalent to*

$$(3) \quad \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty.$$

*If  $V \leq 0$ , then for every  $T \in (0, \infty)$ , (2) is equivalent to*

$$(4) \quad \sup_{0 < t \leq T, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty.$$

It is appropriate to say that  $V$  satisfying (3) or (4) has bounded potential for bridges (is bridges potential-bounded), globally or locally in time, respectively, cf. Section 2.

**Lemma 1.2.** *If for some  $h > 0$  and  $0 \leq \eta < 1$  we have*

$$\sup_{0 < t \leq h, x, y \in \mathbb{R}^d} S(V^+, t, x, y) \leq \eta,$$

*and  $S(V^-)$  is bounded on bounded subsets of  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$  then*

$$(5) \quad e^{-S(V^-, t, x, y)} \leq \frac{G(t, x, y)}{g(t, x, y)} \leq \left( \frac{1}{1-\eta} \right)^{1+t/h}, \quad t > 0, x, y \in \mathbb{R}^d.$$

We record the following observations on integrability and on the potential-boundedness (7) of functions  $V$  which are bridges potential-bounded.

**Lemma 1.3.** *If  $S(V, t, x, y) < \infty$  for some  $t > 0$ ,  $x, y \in \mathbb{R}^d$ , then  $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ . If (4) holds, then*

$$(6) \quad \sup_{x \in \mathbb{R}^d} \int_0^T \int_{\mathbb{R}^d} g(s, x, z) |V(z)| dz ds < \infty.$$

*If (3) even holds, then  $|V|$  has bounded Newtonian potential:*

$$(7) \quad \sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z) |V(z)| dz ds < \infty.$$

*If  $V \geq 0$ , then (1) implies (3) and (2) implies (4). If  $d = 3$  and  $V \leq 0$ , then (7) is equivalent to (1).*

*Proof.* The first statement follows, because  $g(t, x, y)$  is locally bounded from below on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$  (see [12, Lemma 3.7] for a quantitative general result). Since  $\int_{\mathbb{R}^d} S(V, t, x, y)g(t, x, y) dy = \int_0^t \int_{\mathbb{R}^d} g(s, x, z)|V(z)| dz ds$ , we see that (4) implies (6) and (3) implies (7). The next to the last sentence easily follows from Duhamel formula and the fact that  $G \geq g$  in this case. The last statement follows from [19, Remark (2) and (3) on p. 4].  $\square$

We note that (7) and thus also (3) fail for all nonzero  $V$  in dimensions  $d = 1$  and  $d = 2$ , because then  $\int_0^\infty g(s, x, z) ds \equiv \infty$ . Consequently, (1) fails for nontrivial  $V \leq 0$  if  $d = 1$  or  $2$ . For  $d \geq 3$  we let  $C_d = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}$ . The Newtonian potential of nonnegative function  $f$  and  $x \in \mathbb{R}^d$  will be denoted

$$-\Delta^{-1}f(x) := \int_0^\infty \int_{\mathbb{R}^d} g(s, x, z)f(z) dz ds = C_d \int_{\mathbb{R}^d} \frac{1}{|z-x|^{d-2}} f(z) dz.$$

Thus, (7) reads  $\|\Delta^{-1}|V|\|_\infty < \infty$ . We also note that by Theorem 1C (2), Remark (3) on p. 4, and the comments before Theorem 1B in [19],  $\|\Delta^{-1}V\|_\infty < 1$  suffices for (1) when  $d = 3$  and  $V \geq 0$ .

## 2. SUFFICIENT CONDITIONS FOR THE SHARP GAUSSIAN ESTIMATES

Recall from [8, (2.5)] that for  $p \in [1, \infty]$ ,

$$\|P_t f\|_\infty \leq C(d, p) t^{-d/(2p)} \|f\|_p, \quad t > 0,$$

where  $P_t f(x) = \int_{\mathbb{R}^d} g(t, x, z)f(z) dz$ ,  $f \in L^p(\mathbb{R}^d)$  and

$$C(d, p) = \begin{cases} (4\pi)^{-d/2}, & \text{if } p = 1, \\ (4\pi)^{-d/(2p)} (1 - p^{-1})^{(1-p^{-1})d/2}, & \text{if } p \in (1, \infty]. \end{cases}$$

We will give an analogue for the *bridges*  $T_s^{t,y}$ . Here  $t > 0$ ,  $y \in \mathbb{R}^d$ , and

$$T_s^{t,y} f(x) = \int_{\mathbb{R}^d} \frac{g(s, x, z) g(t-s, z, y)}{g(t, x, y)} f(z) dz, \quad 0 < s < t, \quad x \in \mathbb{R}^d.$$

Clearly,

$$(8) \quad T_s^{t,y} f(x) = T_{t-s}^{t,x} f(y), \quad 0 < s < t, \quad x, y \in \mathbb{R}^d.$$

By the Chapman-Kolmogorov equations (the semigroup property) for the kernel  $g$ , we have  $T_s^{t,y} 1 = 1$ . We also note that  $S(V)$  is related to the potential (0-resolvent) operator of  $T$  as follows,

$$S(V, t, x, y) = \int_0^t T_s^{t,y} |V|(x) ds.$$

**Lemma 2.1.** *For  $p \in [1, \infty]$  and  $f \in L^p(\mathbb{R}^d)$  we have*

$$\|T_s^{t,y} f\|_\infty \leq C(d, p) \left[ \frac{(t-s)s}{t} \right]^{-d/(2p)} \|f\|_p, \quad 0 < s < t, \quad y \in \mathbb{R}^d.$$

*Proof.* We note that

$$\frac{g(s, x, z) g(t-s, z, y)}{(4\pi)^{-d/2} g(t, x, y)} = \left[ \frac{(t-s)s}{t} \right]^{-\frac{d}{2}} \exp \left[ -\frac{|z-x|^2}{4s} - \frac{|y-z|^2}{4(t-s)} + \frac{|y-x|^2}{4t} \right].$$

As in [25, (3.4)], we have

$$(9) \quad \frac{|z-x|^2}{4s} + \frac{|y-z|^2}{4(t-s)} \geq \frac{|y-x|^2}{4t}.$$

Indeed, (9) obtains from by the triangle and Cauchy-Schwarz inequalities:

$$|y-x| \leq \sqrt{s} \frac{|z-x|}{\sqrt{s}} + \sqrt{t-s} \frac{|y-z|}{\sqrt{t-s}} \leq \sqrt{t} \left( \frac{|z-x|^2}{s} + \frac{|y-z|^2}{t-s} \right)^{1/2}.$$

For  $p = 1$ , the assertion of the lemma results from (9). For  $p \in (1, \infty)$ , we let  $p' = p/(p-1)$ , apply Hölder's inequality and the semigroup property, and by the first case we obtain

$$\begin{aligned} |T_s^{t,y} f(x)| &\leq g(t, x, y)^{-1} \left[ \int g(s, x, z)^{p'} g(t-s, z, y)^{p'} dz \right]^{1/p'} \|f\|_p \\ &= C(d, p) \left[ \frac{s(t-s)}{t} \right]^{-d/(2p)} \|f\|_p. \end{aligned}$$

Here we also use the identity  $g(s, x, z)^{p'} = g(s/p', x, z)(4\pi s)^{(1-p')d/2}(p')^{-d/2}$ . For  $p = \infty$ , the assertion follows from the identity  $T_s^{t,y} 1 = 1$ .  $\square$

Zhang [27, Proposition 2.1] showed that (1) and (2) hold for  $V$  in specific  $L^p$  spaces (see also [27, Theorem 1.1 and Remark 1.1]). We can reprove his result as follows.

**Proposition 2.2.** *Let  $V: \mathbb{R}^d \rightarrow \mathbb{R}$  and  $p, q \in [1, \infty]$ .*

(a) *If  $V \in L^p(\mathbb{R}^d)$ ,  $p > d/2$  and  $c = C(d, p) \frac{[\Gamma(1-d/(2p))]}{\Gamma(2-d/p)} \|V\|_p$ , then*

$$\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \leq c t^{1-d/(2p)}, \quad t > 0.$$

(b) *If  $V \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$  and  $q < d/2 < p$ , then (3) holds.*

*Proof.* Part (a) follows from Lemma 2.1, so we proceed to (b). For  $t > 2$ ,

$$(10) \quad \int_0^t T_s^{t,y} |V|(x) ds = \int_0^{t/2} T_s^{t,y} |V|(x) ds + \int_0^{t/2} T_s^{t,x} |V|(y) ds.$$

Estimating the first term of the sum, by Lemma 2.1 we obtain

$$\begin{aligned} \int_0^{t/2} T_s^{t,y} |V|(x) ds &\leq c \|V\|_p \int_0^1 \left[ \frac{(t-s)s}{t} \right]^{-d/(2p)} ds + c \|V\|_q \int_1^{t/2} \left[ \frac{(t-s)s}{t} \right]^{-d/(2q)} ds \\ (11) \quad &\leq c' \|V\|_p \int_0^1 s^{-d/(2p)} ds + c' \|V\|_q \int_1^\infty s^{-d/(2q)} ds. \end{aligned}$$

By (8), the second term has the same bound. For  $t \in (0, 2]$  we use (a).  $\square$

By Lemma 1.1 and 1.2 we get the following conclusion.

**Corollary 2.3.** *Under the assumptions of Proposition 2.2(a),  $G$  satisfies the sharp local Gaussian bounds (2). If  $V \leq 0$  and the assumptions of Proposition 2.2(b) hold, then  $G$  has the sharp global Gaussian bounds (1).*

Recall that [16, Theorem 2] and [19, Remark (1) and (4) on p. 4] yield (1) for  $d \geq 4$  if  $\|\Delta^{-1}V^-\|_\infty$  and  $\|V^-\|_{d/2}$  are finite,  $\|\Delta^{-1}V^+\|_\infty < 1$  and  $\|V^-\|_{d/2}$  is small. We can reduce Proposition 2.2(b) to this result as follows.

**Lemma 2.4.** *The assumptions of Proposition 2.2(b) necessitate that  $d \geq 3$ ,  $V \in L^{d/2}(\mathbb{R}^d)$  and  $\|\Delta^{-1}|V|\|_\infty < \infty$ .*

*Proof.* Plainly, the assumptions of Proposition 2.2(b) imply  $d > 2$  and  $V \in L^{d/2}(\mathbb{R}^d)$ . We now verify that  $\|\Delta^{-1}|V|\|_\infty < \infty$ . By Hölder's inequality,

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \int_{B(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} dz &\leq \| |z|^{2-d} \mathbf{1}_{B(0,1)}(z) \|_{p'} \|V\|_p < \infty, \\ \sup_{x \in \mathbb{R}^d} \int_{B^c(0,1)} \frac{|V(z+x)|}{|z|^{d-2}} dz &\leq \| |z|^{2-d} \mathbf{1}_{B^c(0,1)}(z) \|_{q'} \|V\|_q < \infty, \end{aligned}$$

where  $p', q'$  are the exponents conjugate to  $p, q$ , respectively.  $\square$

In what follows, we propose suitable sufficient conditions for (1) and (2). We let  $d_1, d_2 \in \mathbb{N}$  and  $d = d_1 + d_2$ .

**Remark 2.5.** The Gauss-Weierstrass kernel  $g(t, x)$  in  $\mathbb{R}^d$  can be represented as a tensor product:

$$g(t, x) = (4\pi t)^{-d_1/2} e^{-|x_1|^2/(4t)} (4\pi t)^{-d_2/2} e^{-|x_2|^2/(4t)},$$

where  $x_1 \in \mathbb{R}^{d_1}$ ,  $x_2 \in \mathbb{R}^{d_2}$  and  $x = (x_1, x_2)$ . The kernels of the bridges factorize accordingly:

$$\begin{aligned} &\frac{g(s, x, z) g(t-s, z, y)}{g(t, x, y)} \\ &= \frac{(4\pi s)^{-d_1/2} e^{-|z_1-x_1|^2/(4s)} (4\pi(t-s))^{-d_1/2} e^{-|y_1-z_1|^2/(4(t-s))}}{(4\pi t)^{-d_1/2} e^{-|y_1-x_1|^2/(4t)}} \\ &\quad \times \frac{(4\pi s)^{-d_2/2} e^{-|z_2-x_2|^2/(4s)} (4\pi(t-s))^{-d_2/2} e^{-|y_2-z_2|^2/(4(t-s))}}{(4\pi t)^{-d_2/2} e^{-|y_2-x_2|^2/(4t)}}. \end{aligned}$$

**Corollary 2.6.** *Let  $V_1: \mathbb{R}^{d_1} \rightarrow \mathbb{R}$ ,  $V_2: \mathbb{R}^{d_2} \rightarrow \mathbb{R}$ , and  $V(x) = V_1(x_1)V_2(x_2)$ , where  $x = (x_1, x_2) \in \mathbb{R}^d$ ,  $x_1 \in \mathbb{R}^{d_1}$  and  $x_2 \in \mathbb{R}^{d_2}$ . Assume that  $V_1 \in L^\infty(\mathbb{R}^{d_1})$  and  $\sup_{t>0, x_2, y_2 \in \mathbb{R}^{d_2}} S(V_2, t, x_2, y_2) < \infty$ . Then (3) holds.*

*Proof.* In estimating  $S(V, t, x, y)$  we first use the factorization of the bridges and the boundedness of  $V_1$ , and then the Chapman-Kolmogorov equations and the boundedness of  $S(V_2)$ .  $\square$

Let  $p, p_1, p_2 \in [1, \infty]$ .

**Definition 2.7.** *We write  $f \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$  if there are  $f_1 \in L^{p_1}(\mathbb{R}^{d_1})$  and  $f_2 \in L^{p_2}(\mathbb{R}^{d_2})$ , such that*

$$f(x_1, x_2) = f_1(x_1)f_2(x_2), \quad x_1 \in \mathbb{R}^{d_1}, \quad x_2 \in \mathbb{R}^{d_2}.$$

Clearly,  $L^p(\mathbb{R}^{d_1}) \times L^p(\mathbb{R}^{d_2}) \subset L^p(\mathbb{R}^{d_1+d_2})$ , in fact  $\|f\|_p = \|f_1\|_p \|f_2\|_p$  if  $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ .

**Lemma 2.8.** *For  $f(x_1, x_2) = f_1(x_1)f_2(x_2) \in L^{p_1}(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$ ,  $0 < s < t$  and  $y \in \mathbb{R}^d$ , we have*

$$\|T_s^{t,y} f\|_\infty \leq C(d_1, p_1) C(d_2, p_2) \left[ \frac{(t-s)s}{t} \right]^{-d_1/(2p_1)-d_2/(2p_2)} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

*Proof.* We proceed as in the proof of Lemma 2.1, using Remark 2.5.  $\square$

We extend Proposition 2.2 as follows.

**Theorem 2.9.** *Let  $d_1, d_2 \in \mathbb{N}$ ,  $d = d_1 + d_2$ ,  $V: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $p_1, p_2 \in [1, \infty]$  and*

$$\frac{d_1}{2p_1} + \frac{d_2}{2p_2} = 1.$$

(a) *If  $r \in (p_1, \infty]$  and  $V \in L^r(\mathbb{R}^{d_1}) \times L^{p_2}(\mathbb{R}^{d_2})$ , then*

$$\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y) \leq c t^{1-d_1/(2r)-d_2/(2p_2)},$$

$$\text{where } c = C(d_1, r)C(d_2, p_2) \frac{[\Gamma(1-d_1/(2r)-d_2/(2p_2))]^2}{\Gamma(2-d_1/r-d_2/p_2)} \|V_1\|_r \|V_2\|_{p_2}.$$

(b) *If  $1 \leq q < p_1 < r \leq \infty$  and  $V \in [L^q(\mathbb{R}^{d_1}) \cap L^r(\mathbb{R}^{d_1})] \times L^{p_2}(\mathbb{R}^{d_2})$ , then (3) holds.*

*Proof.* We follow the proof of Proposition 2.2, replacing Lemma 2.1 by Lemma 2.8.  $\square$

By Lemma 1.1 and 1.2 we get the following conclusion.

**Corollary 2.10.** *Under the assumptions of Theorem 2.9(a),  $G$  satisfies the sharp local Gaussian bounds (2). If  $V \leq 0$  and the assumptions of Theorem 2.9(b) hold, then  $G$  has the sharp global Gaussian bounds (1).*

Clearly, if  $|U| \leq |V|$ , then  $S(U) \leq S(V)$ . This may be used to extend the conclusions of Theorem 2.9 and Corollary 2.10 beyond tensor products  $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ .

### 3. EXAMPLES

Let  $\mathbf{1}_A$  denote the indicator function of  $A$ . In what follows,  $G$  in (1) is the Schrödinger perturbation of  $g$  by  $V$ .

**Example 3.1.** *Let  $d \geq 3$  and  $1 < p < \infty$ . For  $x_1 \in \mathbb{R}$ ,  $x_2 \in \mathbb{R}^{d-1}$  we let  $V(x_1, x_2) = -|x_1|^{-1/p} \mathbf{1}_{|x_1| < 1} \mathbf{1}_{|x_2| < 1}$ . Then (1) holds but  $V \notin L_{loc}^p(\mathbb{R}^d)$ .*

Indeed,  $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ , where

$$\begin{aligned} V_1(x_1) &= -|x_1|^{-1/p} \mathbf{1}_{|x_1| < 1}, & x_1 &\in \mathbb{R}, \\ V_2(x_2) &= \mathbf{1}_{|x_2| < 1}, & x_2 &\in \mathbb{R}^{d-1}. \end{aligned}$$

Let

$$1 \leq q < p_1 < r < p,$$

and

$$p_2 = \frac{d-1}{2} \frac{p_1}{p_1 - 1/2}.$$

Since  $d \geq 3$ ,  $p_2 > 1$ . In the notation of Theorem 2.9 we have  $d_1 = 1$ ,  $d_2 = d - 1$ , and indeed  $d_1/(2p_1) + d_2/(2p_2) = 1$ . Since  $V_1 \in L^r(\mathbb{R}) \cap L^q(\mathbb{R})$  and  $V_2 \in L^{p_2}(\mathbb{R}^{d-1})$ , the assumptions of Theorem 2.9(b) are satisfied, and (1) follows by Corollary 2.10. Clearly,  $V \notin L_{loc}^p(\mathbb{R}^d)$ .

**Example 3.2.** For  $d \geq 3$ ,  $n = 2, 3, \dots$ , let  $V_n(x) = |x_1|^{-1+1/n} \mathbf{1}_{|x_1| < 1} \mathbf{1}_{|x_2| < 1}$ , where  $x = (x_1, x_2)$ ,  $x_1 \in \mathbb{R}$ ,  $x_2 \in \mathbb{R}^{d-1}$ . Let  $a_n = \sup_{t>0, x, y \in \mathbb{R}^d} S(V_n, t, x, y)$ ,

$$V(x) = - \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{V_n(x)}{a_n}, \quad x \in \mathbb{R}^d.$$

Then (1) holds but  $V \notin \bigcup_{p>1} L_{loc}^p(\mathbb{R}^d)$ .

Indeed,  $0 < a_n < \infty$  by Example 3.1, and so

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) \leq \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

This yields the global sharp Gaussian bounds. For  $p > 1$  we let  $m = \lceil \frac{p}{p-1} \rceil$ , and we have  $m \geq 2$ ,  $\frac{m-1}{m}p \geq 1$ . Then,

$$\int_{B(0,2)} |V(x)|^p dx \geq \left( \frac{1}{m^2 a_m} \right)^p \int_{|x_2| < 1} \int_{|x_1| < 1} |x_1|^{-\frac{m-1}{m}p} dx_1 dx_2 = +\infty.$$

**Example 3.3.** Let  $d \geq 3$  and  $V(x_1, x_2) = \frac{-1}{(|x_2|+1)^3}$  for  $x_1 \in \mathbb{R}^{d-3}$ ,  $x_2 \in \mathbb{R}^3$ . Then (1) holds but  $V \notin L^1(\mathbb{R}^d)$ .

Indeed, we denote  $V_2(x_2) = \frac{-1}{(|x_2|+1)^3}$ ,  $x_2 \in \mathbb{R}^d$ , and by the symmetric rearrangement inequality [15, Chapter 3], in dimension  $d = 3$  we have

$$0 \leq \Delta^{-1} V_2 \leq C_3 \int_{\mathbb{R}^3} \frac{1}{|z|(|z|+1)^3} dz < \infty.$$

By Lemma 1.3 and Lemma 1.1

$$\sup_{t>0, x_2, y_2 \in \mathbb{R}^3} S(V_2, t, x_2, y_2) < \infty.$$

By Corollary 2.6 and Lemma 1.1 we see that (1) holds. Clearly,  $V \notin L^1(\mathbb{R}^d)$ .

**Corollary 3.4.** For every  $d \geq 3$  there is a function  $V$  such that (1) holds but  $V \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L_{loc}^p(\mathbb{R}^d)$ .

*Proof.* Take the sum of the functions from Example 3.2 and Example 3.3.  $\square$

We can have nonnegative examples, too. Namely, let  $V \leq 0$  be as in Corollary 3.4. Then  $M = \sup_{t>0, x, y \in \mathbb{R}^d} S(V, t, x, y) < \infty$ . We let  $\tilde{V} = |V|/(M+1)$ . Then  $\tilde{V} \geq 0$ ,  $\tilde{V} \notin L^1(\mathbb{R}^d) \cup \bigcup_{p>1} L_{loc}^p(\mathbb{R}^d)$  and

$$\sup_{t>0, x, y \in \mathbb{R}^d} S(\tilde{V}, t, x, y) = M/(M+1) < 1.$$

Therefore (5) holds for  $\tilde{V}$  with  $h = \infty$  and  $\eta = M/(M+1)$ , which yields (1).

Let  $d_1, d_2 \in \mathbb{N}$ ,  $d = d_1 + d_2$ ,  $V_1: \mathbb{R}^{d_1} \rightarrow \mathbb{R}$ ,  $V_2: \mathbb{R}^{d_2} \rightarrow \mathbb{R}$ , and  $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ , where  $x_1 \in \mathbb{R}^{d_1}$  and  $x_2 \in \mathbb{R}^{d_2}$ . Let  $G_1(t, x_1, y_1)$ ,  $G_2(t, x_2, y_2)$  be the Schrödinger perturbations of the Gauss-Weierstrass kernels on  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$  by  $V_1$  and  $V_2$ , respectively. Then  $G(t, (x_1, x_2), (y_1, y_2)) := G_1(t, x_1, y_1) + G_2(t, x_2, y_2)$  is the Schrödinger perturbation of the Gauss-Weierstrass kernel on  $\mathbb{R}^d$  by  $V$ . Clearly, if the sharp Gaussian estimates hold for  $G_1$  and  $G_2$ , then they hold for  $G$ . Our next example is aimed to show that such trivial conclusions are invalid for tensor products  $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ .



**Example 3.5.** Let  $\varepsilon \in [0, 1)$ . For  $x_1, x_2 \in \mathbb{R}^3$  let  $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ , where

$$V_1(x) = V_2(x) = -\frac{1-\varepsilon}{2} |x|^{-1-\varepsilon} \mathbf{1}_{|x|<1}.$$

Then the fundamental solutions in  $\mathbb{R}^3$  of  $\partial_t = \Delta + V_1$  and  $\partial_t = \Delta + V_2$  satisfy (1) and (2), but that of  $\partial_t = \Delta + V$  in  $\mathbb{R}^6$  satisfies neither (1) nor (2).

Indeed, by the symmetric rearrangement inequality [15, Chapter 3],

$$0 \leq -\Delta^{-1}V_1(x) \leq -\Delta^{-1}V_1(0) = \frac{1-\varepsilon}{8\pi} \int_{\{z \in \mathbb{R}^3: |z|<1\}} \frac{1}{|z|} |z|^{-1-\varepsilon} dz = 1/2,$$

for all  $x \in \mathbb{R}^3$ . Thus,  $\|\Delta^{-1}V_1\|_\infty = \|\Delta^{-1}V_2\|_\infty < \infty$ . By Lemma 1.3 we get (1) for the fundamental solutions in  $\mathbb{R}^3$  of  $\partial_t = \Delta + V_1$  and  $\partial_t = \Delta + V_2$ . However, the fundamental solution in  $\mathbb{R}^6$  of  $\partial_t = \Delta + V$  fails (2). Indeed, if we let  $T \leq 1$ ,  $a \in \mathbb{R}^6$ ,  $|a| = 1$ , and  $c = \int_0^1 p(s, 0, a) ds$ , then by [9, Lemma 3.5],

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^6} g(s, 0, x) |V(x)| dx ds &\geq \int_{\{x \in \mathbb{R}^6: |x|^2 \leq T\}} \int_0^T g(s, 0, x) ds |V(x)| dx \\ &\geq c \int_{\{x \in \mathbb{R}^6: |x|^2 \leq T\}} \frac{1}{|x|^4} |V(x)| dx \\ &\geq c \int_{\{x_1 \in \mathbb{R}^3: |x_1|^2 < T/2\}} |V_1(x_1)| \int_{\{x_2 \in \mathbb{R}^3: |x_2|^2 < T/2\}} \frac{|V_2(x_2)|}{(|x_1|^2 + |x_2|^2)^2} dx_2 dx_1 \\ &\geq \frac{c(1-\varepsilon)}{2} \int_{\{x_1 \in \mathbb{R}^3: |x_1|^2 < T/2\}} |V_1(x_1)| \int_{\{x_2 \in \mathbb{R}^3: |x_2|^2 < T/2\}} \frac{|x_2|^{-1}}{(|x_1|^2 + |x_2|^2)^2} dx_2 dx_1 \\ &= \frac{c(1-\varepsilon)}{2} \int_{\{x_1 \in \mathbb{R}^3: |x_1|^2 < T/2\}} |V_1(x_1)| \frac{\pi T}{|x_1|^2(T/2 + |x_1|^2)} dx_1 \\ &= \pi^2 c T (1-\varepsilon)^2 \int_0^{\sqrt{T/2}} \frac{r^{-1-\varepsilon}}{T/2 + r^2} dr = \infty. \end{aligned}$$

By Lemma 1.3, (4) fails, and so does (2), cf. Lemma 1.1. Thus, the sharp Gaussian estimates may hold for the Schrödinger perturbations of the Gauss-Weierstrass kernels by  $V_1$  and  $V_2$  but fail for the Schrödinger perturbation of the Gauss-Weierstrass kernel by  $V(x_1, x_2) = V_1(x_1)V_2(x_2)$ . Considering  $-V_1$  and  $-V_2$  above by the last two sentences of Section 1, we can have a similar example for nonnegative perturbations, because  $1/2 < 1$ . Let us also remark that the sharp global Gaussian estimates may hold for  $V(x_1, x_2) = V_1(x_1)V_2(x_2)$  but fail for  $V_1$  or  $V_2$ . Indeed, it suffices to consider  $V_1(x_1) = -\mathbf{1}_{|x_1|<1}$  on  $\mathbb{R}^3$  and  $V_2 \equiv 1$  on  $\mathbb{R}$ , and to apply Theorem 2.9. We see that it is the combined effect of the factors  $V_1$  and  $V_2$  that matters—as captured in Section 2.

#### 4. APPENDIX

Following [5, 7] we study and use the following functions

$$\begin{aligned} f(t) &= \sup_{x, y \in \mathbb{R}^d} S(V, t, x, y), \quad t \in (0, \infty), \\ F(t) &= \sup_{0 < s < t} f(s) = \sup_{\substack{x, y \in \mathbb{R}^d \\ 0 < s < t}} S(V, s, x, y), \quad t \in (0, \infty]. \end{aligned}$$

We fix  $V$  and  $x, y \in \mathbb{R}^d$ . For  $0 < \varepsilon < t$ , we consider

$$S(V, t - \varepsilon, x, y) = \int_0^t \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t - \varepsilon - s, z, y)}{g(t - \varepsilon, x, y)} |V(z)| \mathbf{1}_{[0, t - \varepsilon]}(u) dz ds.$$

By Fatou's lemma we get

$$S(V, t, x, y) \leq \liminf_{\varepsilon \rightarrow 0} S(V, t - \varepsilon, x, y),$$

meaning that  $(0, \infty) \ni t \mapsto S(V, t, x, y)$  is lower semicontinuous on the left. It follows that  $f$  is lower semi-continuous on the left, too. In consequence,  $f(t) \leq F(t)$  and  $F(t) = \sup_{0 < s \leq t} f(s)$  for  $0 < t < \infty$ .

We next claim that  $f$  is sub-additive, that is,

$$(12) \quad f(t_1 + t_2) \leq f(t_1) + f(t_2), \quad t_1, t_2 > 0.$$

This follows from the Chapman-Kolmogorov equations for  $g$ . Indeed, we have  $S(V, t_1 + t_2, x, y) = I_1 + I_2$ , where

$$\begin{aligned} I_1 &= \int_0^{t_1} \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t_1 + t_2 - s, z, y)}{g(t_1 + t_2, x, y)} |V(z)| dz ds \\ &= \int_0^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t_1 - s, z, w)g(t_2, w, y)g(t_1, x, w)}{g(t_1 + t_2, x, y)g(t_1, x, w)} |V(z)| dw dz ds \\ &\leq \int_{\mathbb{R}^d} \frac{g(t_2, w, y)g(t_1, x, w)}{g(t_1 + t_2, x, y)} S(V, t_1, x, w) dw \leq f(t_1), \end{aligned}$$

and  $I_2$  equals

$$\begin{aligned} &\int_{t_1}^{t_1+t_2} \int_{\mathbb{R}^d} \frac{g(s, x, z)g(t_1 + t_2 - s, z, y)}{g(t_1 + t_2, x, y)} |V(z)| dz ds \\ &= \int_{t_1}^{t_1+t_2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{g(t_1, x, w)g(s - t_1, w, z)g(t_2 - (s - t_1), z, y)g(t_2, w, y)}{g(t_1 + t_2, x, y)g(t_2, w, y)} |V(z)| dw dz ds \\ &\leq \int_{\mathbb{R}^d} \frac{g(t_1, x, w)g(t_2, w, y)}{g(t_1 + t_2, x, y)} S(V, t_2, w, y) dw \leq f(t_2). \end{aligned}$$

This yields (12).

**Lemma 4.1.** *For all  $t, h > 0$  we have  $f(t) \leq F(h) + t f(h)/h$ .*

*Proof.* Let  $k \in \mathbb{N}$  be such that  $(k - 1)h < t \leq kh$ , and let  $\theta = t - (k - 1)h$ . Then  $t = \theta + (k - 1)h$ , and by (12) we get

$$f(t) \leq f(\theta) + t f(h)/h \leq F(h) + t f(h)/h,$$

since  $0 < \theta \leq h$ . □

**Corollary 4.2.**  *$F(t) \leq F(h) + t F(h)/h$  and  $F(2t) \leq 2F(t)$  for  $t, h > 0$ .*

*Proof of Lemma 1.2.* The left-hand side of (5) follows from [27, pp. 467-469] and Lemma 1.3, or we can use [5, (41)], which follows therein from Jensen's inequality and the second displayed formula on page 252 of [5]. We now prove the right hand side of (5). Since  $G$  is increasing in  $V$ , we may assume that  $V \geq 0$ . For  $0 < s < t$ ,  $x, y \in \mathbb{R}^d$ , we let  $p_0(s, x, t, y) = g(t - s, x, y)$  and  $p_n(s, x, t, y) = \int_s^t \int_{\mathbb{R}^d} p_{n-1}(s, x, u, z) V(z) p_0(u, z, t, y) dz du$ ,  $n \in \mathbb{N}$ . Let

$Q : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  satisfy  $Q(u, r) + Q(r, v) \leq Q(u, v)$ . By [14, Theorem 1] (see also [6, Theorem 3]) if there is  $0 < \eta < 1$  such that

$$(13) \quad p_1(s, x, t, y) \leq [\eta + Q(s, t)]p_0(s, x, t, y),$$

then

$$(14) \quad \tilde{p}(s, x, t, y) := \sum_{n=0}^{\infty} p_n(s, x, t, y) \leq \left( \frac{1}{1-\eta} \right)^{1+\frac{Q(s,t)}{\eta}} p_0(s, x, t, y).$$

Corollary 4.2 and the assumptions of the lemma imply that (13) is satisfied with  $\eta = F(h) < 1$  and  $Q(s, t) = (t - s)F(h)/h$ . Since  $G(t, x, y) = \tilde{p}(0, x, t, y)$ , the proof of (5) is complete (see also [5, (17)]).  $\square$

*Proof of Lemma 1.1.* Let  $V \leq 0$ . By the proof of Theorem 1.1(a) at the bottom of p. 468 in [27], the boundedness of  $S(V, t, x, y)$  is necessary and sufficient for (1). In particular, by the displayed formula proceeding [27, (3.1)] the boundedness of  $S(V, t, x, y)$  is sufficient for (1). Alternatively we can apply Jensen inequality to the second displayed formula on p. 252 in [5]. The first part of Lemma 1.1 is proved. The second part is obtained in the same way, by restricting the considerations, and the transition kernel, to bounded time interval.  $\square$

As a consequence of Corollary 4.2 we obtain the following result.

**Corollary 4.3.** *Let  $V \leq 0$  and  $T > 0$ . Then (2) holds if and only if*

$$(15) \quad Ce^{-ct}g(t, x, y) \leq G(t, x, y), \quad t > 0, x, y \in \mathbb{R}^d,$$

for some constants  $C > 0, c \geq 0$ . In fact we can take

$$\ln C = - \sup_{\substack{x, y \in \mathbb{R}^d \\ 0 < t \leq T}} S(V, t, x, y) \quad \text{and} \quad c = \frac{1}{T} \sup_{x, y \in \mathbb{R}^d} S(V, T, x, y).$$

*Proof.* Obviously, (15) implies (2) for every fixed  $T > 0$ . Conversely, if (2) holds for fixed  $T > 0$ , then by Lemma 1.2 and 4.1 we have

$$\frac{G(t, x, y)}{g(t, x, y)} \geq e^{-S(V, t, x, y)} \geq e^{-f(t)} \geq e^{-F(T)} e^{-tf(T)/T}.$$

$\square$

We note in passing that the above proof shows that (2) is determined by the behavior of  $\sup_{x, y \in \mathbb{R}^d} S(V, t, x, y)$  for small  $t > 0$ . We end our discussion by recalling the connection of  $G$  to  $\Delta + V$  aforementioned in Abstract. As it is well known, and can be directly checked by using the Fourier transform or by arguments of the semigroup theory [3, Section 4],

$$\int_s^\infty \int_{\mathbb{R}^d} g(u - s, x, z) \left[ \partial_u \phi(u, z) + \Delta \phi(u, z) \right] dz du = -\phi(s, x),$$

for all  $s \in \mathbb{R}, x \in \mathbb{R}^d$  and for all  $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ , the smooth compactly supported test functions on space-time. Similarly, if  $V$  satisfies the assumptions of Lemma 1.2, then by [27, Theorem 1.1] for all  $s \in \mathbb{R}, x \in \mathbb{R}^d, \phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ ,

$$\int_s^\infty \int_{\mathbb{R}^d} G(u - s, x, z) \left[ \partial_u \phi(u, z) + \Delta \phi(u, z) + V(z)\phi(u, z) \right] dz du = -\phi(s, x).$$

We refer to [6, Lemma 4] for a general approach to such identities.

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WROCLAW UNIVERSITY OF SCIENCE AND TECHNOLOGY, WYBRZEŻE WYSPIAŃSKIEGO  
27, 50-370 WROCLAW, POLAND  
*E-mail address:* bogdan@pwr.edu.pl

UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND  
*E-mail address:* Jacek.Dziubanski@math.uni.wroc.pl

UNIVERSITÄT BIELEFELD, POSTFACH 10 01 31, D-33501 BIELEFELD, GERMANY AND  
WROCLAW UNIVERSITY OF SCIENCE AND TECHNOLOGY, WYBRZEŻE WYSPIAŃSKIEGO  
27, 50-370 WROCLAW, POLAND  
*E-mail address:* karol.szczypkowski@math.uni-bielefeld.de, karol.szczypkowski@pwr.edu.pl